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On six-dimensional pseudo-Riemannian almost g.o. spaces

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Abstract

We modify the "Kaplan example" (a six-dimensional nilpotent Lie group which is a Riemannian g.o. space) and we obtain two pseudo-Riemannian homogeneous spaces with noncompact isotropy group. These examples have the property that all geodesics are homogeneous up to a set of measure zero. We also show that the (incomplete) geodesic graphs are strongly discontinuous at the boundary, i.e., the limits along certain curves are always infinite.

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1. Introduction

Homogeneous geodesics on homogeneous Riemannian manifolds were studied for example in [11–13]. Further references can be found also in [4]. In physics, Penrose limits along null homogeneous geodesics are studied in [7,15]. In [15], it is shown that the Penrose limit of a Lorentzian spacetime along a homogeneous geodesic is a homogeneous plane wave and the Penrose limit of a reductive homogeneous spacetime along a homogeneous geodesic is a reductive homogeneous plane wave. Null homogeneous geodesics on Lorentzian homogeneous spaces are also studied in [14]. In mathematics, the first results for homogeneous geodesics on pseudo-Riemannian homogeneous manifolds were obtained in [1,4,5]. In [4], Lemma 1.2 is proved and the role of the parameter k is illustrated. In [1], the authors study homogeneous geodesics on three-dimensional Lie groups with Lorentzian metrics. In [5], the pseudo-Riemannian g.o. spaces with compact isotropy group are studied.

Let M be a pseudo-Riemannian manifold. If there is a connected Lie group $G \subset I_0(M)$ which acts transitively on M as a group of isometries, then M is called a *homogeneous pseudo-Riemannian manifold*. Let $p \in M$ be a fixed point. If we denote by H the isotropy group at p, then M can be identified with the *homogeneous space* G/H. In general, there may exist more than one such group $G \subset I_0(M)$. For any fixed choice $M = G/H$, G acts effectively

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on G/H from the left. The pseudo-Riemannian metric g on M can be considered as a G-invariant metric on G/H. The pair $(G/H, g)$ is then called a *pseudo-Riemannian homogeneous space*.

If the metric g is positive definite, then $(G/H, g)$ is always a *reductive* homogeneous space in the following sense: we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively and consider the adjoint representation $\mathrm{Ad}\colon H \times \mathfrak{g} \to \mathfrak{g}$ of H on $\mathfrak{g}$. There exists a direct sum decomposition (*reductive decomposition*) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$. If the metric g is indefinite, the reductive decomposition may not exist (see for instance [7] for the example of nonreductive pseudo-Riemannian homogeneous space). For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_e G$ with the tangent space $T_p M$ via the projection $\pi\colon G \to G/H = M$. Using this natural identification and the scalar product g_p on $T_p M$, we obtain a scalar product $\langle\, , \rangle$ on $\mathfrak{m}$. This scalar product is obviously $\mathrm{Ad}(H)$-invariant.

The definition of a homogeneous geodesic is well known in the Riemannian case (see, e.g., [12]). In the pseudo-Riemannian case, the necessary generalized version was given in [4]:

Definition 1.1. Let $M = G/H$ be a reductive homogeneous pseudo-Riemannian space, $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition and p the basic point of G/H. The geodesic $\gamma(s)$ through the point p defined in an open interval J (where s is an affine parameter) is said to be homogeneous if there exists

(1) a diffeomorphism $s = \varphi(t)$ between the real line and the open interval J;
(2) a vector $X \in \mathfrak{g}$ such that $\gamma(\varphi(t)) = \exp(tX)(p)$ for all $t \in (-\infty, +\infty)$.

The vector X is then called a geodesic vector.

The basic formula characterizing geodesic vectors in the pseudo-Riemannian case appeared in [7,15], but without a proof. The correct mathematical formulation with the proof was given in [4]:

Lemma 1.2. *Let $M = G/H$ be a reductive homogeneous pseudo-Riemannian space, $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition and p the basic point of G/H. Let $X \in \mathfrak{g}$. Then the curve $\gamma(t) = \exp(tX)(p)$ (the orbit of a one-parameter group of isometries) is a geodesic curve with respect to some parameter s if and only if*

$$\langle [X, Z]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = k \langle X_{\mathfrak{m}}, Z \rangle \quad \text{for all } Z \in \mathfrak{m}, \text{ where } k \in \mathbb{R} \text{ is some constant}. \tag{1}$$

Further, if $k = 0$, then t is an affine parameter for this geodesic. If $k \neq 0$, then $s = \mathrm{e}^{-kt}$ is an affine parameter for the geodesic. The second case can occur only if the curve $\gamma(t)$ is a null curve in a (properly) pseudo-Riemannian space.

Definition 1.3. A pseudo-Riemannian homogeneous space $(G/H, g)$ is called a g.o. space if every geodesic of $(G/H, g)$ is homogeneous. Here “g.o.” means “geodesics are orbits”.

It is well known that all *naturally reductive* homogeneous spaces are g.o. spaces. Some decades ago, it was generally believed that also every g.o. space is naturally reductive. The first counter-example of a g.o. space which is in no way naturally reductive comes from Kaplan [9]. This is a six-dimensional Riemannian nilmanifold with a two-dimensional center, one of the so-called “generalized Heisenberg groups”. The extensive study of (Riemannian) g.o. spaces started just with Kaplan’s paper. In the present paper, we are going to consider indefinite metrics on the same six-dimensional manifold.

Our technique used for the characterization of g.o. spaces and g.o. manifolds is based on the concept of “geodesic graph”. The original idea (not using any explicit name) comes from Szenthe [16].

Definition 1.4. Let $(G/H, g)$ be a reductive g.o. space and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ an $\mathrm{Ad}(H)$-invariant decomposition of the Lie algebra $\mathfrak{g}$. A geodesic graph is an $\mathrm{Ad}(H)$-equivariant map $\eta\colon \mathfrak{m} \to \mathfrak{h}$ which is rational on an open dense subset U of $\mathfrak{m}$ and such that $X + \eta(X)$ is a geodesic vector for each $X \in \mathfrak{m}$.

On every reductive g.o. space $(G/H, g)$, there exists at least one geodesic graph. The construction of a *canonical geodesic graph* and *general geodesic graphs* (on open dense subsets) through rational maps is described in detail in [3,10]. On the subset $U \subset \mathfrak{m}$, with respect to a basis $\{E_1, \ldots, E_n\}$ of $\mathfrak{m}$ and a basis $\{F_1, \ldots, F_h\}$ of $\mathfrak{h}$, the components η_i of a geodesic graph are always rational functions in the form $\eta_i = P_i/P$, where P_i and P are homogeneous polynomials (of the coordinates on $T_p(M)$) and $\deg(P_j) = \deg(P) + 1$. For the vectors $X \in \mathfrak{m} \setminus U$, the map η must be constructed part by part using again some rational maps and the geodesic graph may be discontinuous on some subset of $V = \mathfrak{m} \setminus U$. For the examples of geodesic graphs of various degrees we refer the reader to [2,3,6,10]. The systematic description of contemporary results for Riemannian g.o. manifolds was given in [6].

The study of pseudo-Riemannian g.o. spaces started with the paper [5]. In this paper, the present authors considered five-dimensional, six-dimensional and seven-dimensional manifolds which were described with Riemannian metrics in [6,8,12] and geodesic graphs of these Riemannian g.o. manifolds were described in [6,10] (Kaplan's example was one of them). They modified the metrics and obtained pseudo-Riemannian homogeneous spaces with *compact* isotropy group. They showed that these spaces are g.o. spaces and they described the discontinuities of geodesic graphs. At all points of discontinuity, there are different limits along different curves, but these limits are all finite.

In the present paper we modify the Riemannian metrics on the six-dimensional Kaplan's g.o. manifold and we obtain homogeneous pseudo-Riemannian manifolds with *noncompact* isotropy group. We describe the (incomplete) geodesic graphs on open dense subset U of $\mathfrak{m}$, but we show that these pseudo-Riemannian homogeneous manifolds are *not* g.o. manifolds. We also show that, at points $\mathring{X} \in \mathfrak{m} \setminus U$, the limits of geodesic graph for $X \in U$ approaching to $\mathring{X}$ are infinite.

2. Six-dimensional nilpotent example

We are going to study now the six-dimensional example which is a pseudo-Riemannian modification of the Riemannian g.o. space of A. Kaplan. This construction was proposed by P. Meessen in his private correspondence. In this paper, we systematically treat his proposals and solve some of the open questions which he put. In particular, we show the example of a homogeneous space which is not g.o., but all null geodesics are homogeneous.

We describe two pseudo-Riemannian metrics with different signatures on the same manifold. We will treat them simultaneously, because a lot of computations at the Lie algebra level coincide.

2.1. Basic definitions

Let us consider the six-dimensional vector space $\mathfrak{n}$ with the pseudo-orthonormal basis $\{E_1, \ldots, E_4, Z_1, Z_2\}$ with the signature $(-1, -1, 1, 1, \varepsilon, 1)$, where $\varepsilon = \pm 1$.

Let us define the Lie bracket on $\mathfrak{n}$ by the relations

$$\begin{aligned} &[E_1, E_2] = 0, \qquad [E_2, E_3] = Z_2, \\ &[E_1, E_3] = Z_1, \qquad [E_2, E_4] = -Z_1, \\ &[E_1, E_4] = Z_2, \qquad [E_3, E_4] = 0, \qquad (2) \\ &[Z_1, E_i] = [Z_2, E_i] = [Z_1, Z_2] = 0 \quad \text{for } i = 1, \ldots, 4. \qquad (3) \end{aligned}$$

We denote by N the unique connected and simply connected Lie group whose Lie algebra is $\mathfrak{n}$. Further, we denote by A_{ij} (for $1 \le i < j \le 4$) and B_{12} the endomorphisms of $\mathfrak{n}$, with the corresponding action given by the formulas

$$\begin{aligned} &A_{ij}(E_k) = \delta_{ik} E_j - \delta_{jk} E_i \quad \text{for } k = 1, \ldots, 4 \\ &B_{12}(Z_k) = \delta_{1k} Z_2 - \delta_{2k} Z_1 \end{aligned}$$

and we denote by $\bar{A}_{ij}$ (for $1 \le i < j \le 4$) the endomorphisms of $\mathfrak{n}$, with the corresponding action given by the formulas

$$\bar{A}_{ij}(E_k) = \delta_{ik} E_j + \delta_{jk} E_i \quad \text{for } k = 1, \ldots, 4.$$

Further, we define

$$\begin{aligned} &A = A_{34} - A_{12}, \qquad B = \bar{A}_{13} + \bar{A}_{24}, \qquad C = \bar{A}_{14} - \bar{A}_{23}, \\ &\tilde{A} = A_{34} + A_{12} + 2B_{12}. \qquad (4) \end{aligned}$$

The Lie bracket of the operators A, B, C satisfy the relations

$$[A, B] = 2C, \qquad [B, C] = -2A, \qquad [C, A] = 2B \qquad (5)$$

and the operator $\tilde{A}$ commutes with A, B, C. We obtain the isomorphisms $\mathfrak{h} = \operatorname{span}(A, B, C, \tilde{A}) \simeq \mathfrak{so}(1, 2) + \mathfrak{so}(2)$ and $\mathfrak{h}' = \operatorname{span}(A, B, C) \simeq \mathfrak{so}(1, 2)$. We choose $H = \mathrm{SO}(1, 2) \times \mathrm{SO}(2)$ and $H' = \mathrm{SO}(1, 2)$. For $\varepsilon = 1$, it is easy to verify that the algebra $\mathfrak{h}$ acts on $\mathfrak{n}$ by derivations and the scalar product on $\mathfrak{n}$ is invariant with respect to this action. Hence, the manifold N can be expressed as homogeneous space G/H, where $G = N \rtimes H$. For $\varepsilon = -1$, only the smaller algebra $\mathfrak{h}'$ acts invariantly by derivations and hence we can write $N = G'/H'$, where $G' = N \rtimes H'$.

2.2. Geodesic graph on $U \subset \mathfrak{m}$

We are going to construct the canonical geodesic graph. In this case, the symbol ξ is used instead of η. We write a general vector $X \in \mathfrak{m}$ in the form $X = x_1E_1 + \cdots + x_4E_4 + z_1Z_1 + z_2Z_2$ and the vector $\xi(X) \in \mathfrak{h}$ in the form $\xi(X) = \xi_1 A + \xi_2 B + \xi_3 C + \xi_4 \tilde{A}$. Hence we can identify every vector $X \in \mathfrak{m}$ with the arithmetic vector $(x_1, \dots, x_4, z_1, z_2)$ of components with respect to the basis of $\mathfrak{m}$ and every vector $\xi(X) \in \mathfrak{h}$ with the arithmetic vector $(\xi_1, \dots, \xi_4)$ of components with respect to the basis of $\mathfrak{h}$. We use the Lemma 1.2, where we write the vector $X + \xi(X)$ instead of the intended geodesic vector X. We obtain a system of equations for ξ_k depending on x_i and z_j. For $\varepsilon = 1$, the matrix $\mathbf{A}_1$ and the right-hand side vector $\mathbf{b}_1$ of this system are

$$\mathbf{A}_1 = \begin{pmatrix} x_2 & x_3 & x_4 & -x_2 \\ -x_1 & x_4 & -x_3 & x_1 \\ x_4 & -x_1 & x_2 & x_4 \\ -x_3 & -x_2 & -x_1 & -x_3 \\ 0 & 0 & 0 & 2z_2 \\ 0 & 0 & 0 & -2z_1 \end{pmatrix}, \qquad \mathbf{b}_1 = \begin{pmatrix} x_3z_1 + x_4z_2 - kx_1 \\ -x_4z_1 + x_3z_2 - kx_2 \\ -x_1z_1 - x_2z_2 + kx_3 \\ x_2z_1 - x_1z_2 + kx_4 \\ kz_1 \\ kz_2 \end{pmatrix}. \tag{6}$$

For $\varepsilon = -1$ and $\xi(X) \in \mathfrak{h}'$, we write $\xi(X) = \xi_1 A + \xi_2 B + \xi_3 C$ and we identify the vector $\xi(X)$ with the arithmetic vector (ξ_1, ξ_2, ξ_3). The matrix $\mathbf{A}_{-1}$ and the right-hand side vector $\mathbf{b}_{-1}$ of the system of equations given by the Lemma 1.2 are

$$\mathbf{A}_{-1} = \begin{pmatrix} x_2 & x_3 & x_4 \\ -x_1 & x_4 & -x_3 \\ x_4 & -x_1 & x_2 \\ -x_3 & -x_2 & -x_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{b}_{-1} = \begin{pmatrix} -x_3z_1 + x_4z_2 - kx_1 \\ x_4z_1 + x_3z_2 - kx_2 \\ x_1z_1 - x_2z_2 + kx_3 \\ -x_2z_1 - x_1z_2 + kx_4 \\ -kz_1 \\ kz_2 \end{pmatrix}. \tag{7}$$

In both cases $\varepsilon = \pm 1$, if $z_1 = z_2 = 0$ or $x_1 = x_2 = x_3 = x_4 = 0$, we can put $k = 0$ and $\xi(X) = 0$. In this and the next subsection, if not stated otherwise, we suppose that at least one of the x_i and at least one of the z_j are nonzero.

For $\varepsilon = 1$, we see immediately (from the fifth and sixth row) that $\xi_4 = 0$ and $k = 0$. For $\varepsilon = -1$ we see also that $k = 0$. Hence, in both cases $\varepsilon = \pm 1$, we can restrict ourselves on the subalgebra $\mathfrak{h}' = \operatorname{span}(A, B, C)$ and the corresponding restricted system given by the matrix and the right-hand side vector

$$\mathbf{A}' = \begin{pmatrix} x_2 & x_3 & x_4 \\ -x_1 & x_4 & -x_3 \\ x_4 & -x_1 & x_2 \\ -x_3 & -x_2 & -x_1 \end{pmatrix}, \qquad \mathbf{b}' = \begin{pmatrix} \varepsilon x_3z_1 + x_4z_2 \\ -\varepsilon x_4z_1 + x_3z_2 \\ -\varepsilon x_1z_1 - x_2z_2 \\ \varepsilon x_2z_1 - x_1z_2 \end{pmatrix}. \tag{8}$$

The rank of this system is equal to 3 (in the generic case) and by Cramer's rule we obtain the components of the vector $\xi(X)$ in the form

$$\begin{aligned} \xi_1 &= \frac{2\varepsilon z_1(x_1x_4 + x_2x_3) - 2z_2(x_1x_3 - x_2x_4)}{-\|x\|^2}, \\ \xi_2 &= \frac{\varepsilon z_1(x_1^2 - x_2^2 - x_3^2 + x_4^2) + 2z_2(x_1x_2 - x_3x_4)}{-\|x\|^2}, \\ \xi_3 &= \frac{-2\varepsilon z_1(x_1x_2 + x_3x_4) + z_2(x_1^2 - x_2^2 + x_3^2 - x_4^2)}{-\|x\|^2}. \end{aligned} \tag{9}$$

Here we put $\|x\|^2 = x_3^2 + x_4^2 - x_1^2 - x_2^2$. The above formulas describe the geodesic graph on $U = \{X \in \mathfrak{m};\ \|x\|^2 \neq 0\}$.

Remark. In accordance with the general conjecture from [3], $\|x\|^2$ is an invariant with respect to the transformation group $\mathrm{Ad}(H')$.

Further, we put $V = \mathfrak{m} \setminus U = \{X \in \mathfrak{m};\ \|x\|^2 = 0\}$. In the following, we are going to extend the geodesic graph also to some part of V. We remark already here that, in both cases $\varepsilon = \pm 1$, the null-cone N has a nonempty intersection with both U and V.

2.3. Properties of the set $V = \mathfrak{m} \setminus U$

Theorem 2.1. *The subset $V = \mathfrak{m} \setminus U$ of $\mathfrak{m}$ can be decomposed as $V = V_0 + V_1$, where V_0 is an open dense subset of V. The geodesic graph can be defined on V_1 and it cannot be defined on V_0. In particular, because V_0 is nonempty, G/H is not a g.o. space (for each $\varepsilon = \pm 1$).*

Proof. A vector $X \in \mathfrak{m}$ belongs to V if and only if $x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0$. For these vectors, the formulas (9) do not make sense. We have shown in the previous section that for $z_1 = z_2 = 0$, or $x_1 = x_2 = x_3 = x_4 = 0$, a geodesic graph can be defined by $\xi(X) = 0$. Hence we consider now the opposite case, where $X = \sum x_i E_i + \sum z_j Z_j \in V$ and both summands are nonzero.

Clearly, every subdeterminant of rank 3 of the matrix $\mathbf{A}'$ is zero (because it is a multiple of $\|x\|^2$) and $\mathrm{rank}(\mathbf{A}') \leq 2$. Hence, according to the Frobenius criterion of compatibility, the system (8) can be solvable only if all subdeterminants of rank 3 of the extended matrix $\tilde{\mathbf{A}} = (\mathbf{A}'|\mathbf{b}')$ are zero. The significant factors of these subdeterminants are the numerators in the formulas (9). If some of these numerators is nonzero, we obtain from our compatibility conditions that $x_1 = x_2 = x_3 = x_4 = 0$, which is not the case.

On the other hand, the numerators in the formulas (9) are all zero if and only if:

(a) $\varepsilon \frac{z_1}{z_2} = \frac{x_1x_3 - x_2x_4}{x_1x_4 + x_2x_3} = \frac{2(x_3x_4 - x_1x_2)}{x_1^2 - x_2^2 - x_3^2 + x_4^2} = \frac{x_1^2 - x_2^2 + x_3^2 - x_4^2}{2(x_1x_2 + x_3x_4)}$ for $z_2 \neq 0$,

(b) $\varepsilon \frac{z_2}{z_1} = \frac{x_1x_4 + x_2x_3}{x_1x_3 - x_2x_4} = \frac{x_1^2 - x_2^2 - x_3^2 + x_4^2}{2(x_3x_4 - x_1x_2)} = \frac{2(x_1x_2 + x_3x_4)}{x_1^2 - x_2^2 + x_3^2 - x_4^2}$ for $z_1 \neq 0$.

We check easily that the second and the third equality (in both cases) are consequences of the relation $\|x\|^2 = 0$. Hence, all numerators in the formulas (9) are zero if and only if the relation

$$\varepsilon z_1(x_1x_4 + x_2x_3) - z_2(x_1x_3 - x_2x_4) = 0 \tag{10}$$

is valid. For the future convenience, we denote by k_2 and k_3 the polynomials

$$k_2 = x_1x_4 + x_2x_3, \qquad k_3 = x_1x_3 - x_2x_4. \tag{11}$$

Using Cramer's rule and the condition $\|x\|^2 = 0$ we see that the polynomials k_2 and k_3 cannot be both equal to zero.

For a vector $X \in \mathfrak{m}$ such that the relation (10) does not hold, the system (8) (and also the system (6), or (7), respectively) is unsolvable because it does not satisfy the Frobenius criterion of compatibility. Let us define the subsets V_1 and V_0 of V by the relations

$$\begin{aligned} V_1 &= \{X \in \mathfrak{m};\ \|x\|^2 = 0,\ \varepsilon z_1(x_1x_4 + x_2x_3) - z_2(x_1x_3 - x_2x_4) = 0\}, \\ V_0 &= \{X \in \mathfrak{m};\ \|x\|^2 = 0,\ \varepsilon z_1(x_1x_4 + x_2x_3) - z_2(x_1x_3 - x_2x_4) \neq 0\}. \end{aligned} \tag{12}$$

We see that V_1 is just the subset of V for which the geodesic graph can be defined (including the case $z_1 = z_2 = 0$ or $x_1 = x_2 = x_3 = x_4 = 0$). For vectors from V_0, geodesic graph cannot be defined. The set V_0 is obviously open and dense in V. This proves that G/H is not a g.o. space in any case $\varepsilon = \pm 1$. □

In [14], Meessen introduced the definition of a *n.g.o. space*: it is a pseudo-Riemannian homogeneous space, whose any null geodesic is homogeneous. In particular, if the given space is not g.o., he calls it a *proper n.g.o. space*. Our Theorem 2.1 and the first part of the following Theorem 2.2 show, that our example for $\varepsilon = 1$ is a proper n.g.o. space. The geodesic graph can be defined on an open dense subset which contains the null-cone. Geodesic graph is nonlinear and k is zero on the null-cone.

Further, because the geodesic graph on our n.g.o. space is nonlinear on the null-cone, null geodesics are not canonically homogeneous and according to [14], G/H does not admit a $\mathcal{T}_1 \oplus \mathcal{T}_3$-structure. Hence, our pseudo-Riemannian n.g.o. space has the properties required in the open question stated in [14].

Theorem 2.2. *Let us denote by N the null-cone in $\mathfrak{m}$. For $\varepsilon = 1$, it holds $N \cap V = N \cap V_1$. In particular, all null geodesics are homogeneous. For $\varepsilon = -1$, the set $N \cap V_0$ is nonempty and there are null geodesics which are not homogeneous.*

Proof. For $\varepsilon = 1$, the vector $X \in \mathfrak{m}$ is null, if $\|X\|^2 = -x_1^2 - x_2^2 + x_3^2 + x_4^2 + z_1^2 + z_2^2 = 0$. Hence, a null vector X is in U if and only if $z_1^2 + z_2^2 \neq 0$. A null vector X is in V if and only if $z_1 = z_2 = 0$, but in this case we have $\xi(X) = 0$. We see that $N \cap V = N \cap V_1$. In other words, for all null vectors, a geodesic graph is defined.

For $\varepsilon = -1$, the vector $X \in \mathfrak{m}$ is null, if $\|X\|^2 = -x_1^2 - x_2^2 + x_3^2 + x_4^2 - z_1^2 + z_2^2 = 0$. Hence, a null vector X is in U if and only if $z_1 \neq \pm z_2$. It is in V if and only if $z_1 = \pm z_2$. For example, a vector $X = (1, 0, 1, 0, 1, 1) \in \mathfrak{m}$ belongs to $N \cap V_0$. □

Let us now construct a geodesic graph on the subset V_1. Here $\|x\|^2 = 0$ and the relation (10) is valid. The system (8), after omitting unnecessary rows, becomes

$$\mathbf{A}' = \begin{pmatrix} x_2 & x_3 & x_4 \\ -x_1 & x_4 & -x_3 \end{pmatrix}, \qquad \mathbf{b}' = \begin{pmatrix} \frac{z_2}{k_2} x_1(x_3^2 + x_4^2) \\ \frac{z_2}{k_2} x_2(x_3^2 + x_4^2) \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon z_1}{k_3} x_1(x_3^2 + x_4^2) \\ \frac{\varepsilon z_1}{k_3} x_2(x_3^2 + x_4^2) \end{pmatrix}. \tag{13}$$

Here the two right-hand sides correspond to the choice whether we express z_1 or z_2 from the Eq. (10). According to the above remark, at any point of V_1, at least one of these two expressions makes sense. To construct the canonical geodesic graph, we define the subalgebras $\mathfrak{q}_X$ of the isotropy algebra $\mathfrak{h}'$ by the formula

$$\mathfrak{q}_X = \{A \in \mathfrak{h}' \mid [A, X] = 0\} \tag{14}$$

(see for example [10] or [3] for the details of the construction of the canonical geodesic graph when the algebra $\mathfrak{q}_X$ is nontrivial). We have $\dim \mathfrak{q}_X = 1$ (in $\mathfrak{h}'$) and the components q_k of the generator Q_X with respect to the basis $\{A, B, C\}$ are

$$\begin{aligned} q_1 &= x_3^2 + x_4^2, \\ q_2 &= x_1 x_4 - x_2 x_3, \\ q_3 &= -x_1 x_3 - x_2 x_4. \end{aligned} \tag{15}$$

We use the system of equations corresponding to (13) and the condition $\xi(X) \perp Q_X$ (with respect to some invariant scalar product on $\mathfrak{h}'$). By Cramer's rule we obtain components of the canonical geodesic graph in the form

$$\begin{aligned} \xi_1 &= 0, \\ \xi_2 &= \frac{z_2}{k_2}(x_1 x_3 + x_2 x_4) = \frac{\varepsilon z_1}{k_3}(x_1 x_3 + x_2 x_4), \\ \xi_3 &= \frac{z_2}{k_2}(x_1 x_4 - x_2 x_3) = \frac{\varepsilon z_1}{k_3}(x_1 x_4 - x_2 x_3). \end{aligned} \tag{16}$$

Again, at any point of V_1, at least one of the expressions for the solution makes sense. If both expressions make sense, they are equal due to the Eq. (10).

2.4. Limits of $\xi(X)$ for $X \in U$ approaching a value $\mathring{X} \in V$

Theorem 2.3. *For any vector $\mathring{X} \in V$, there is a curve $\gamma(t)$ with the values in $\mathfrak{m}$ and defined on an interval $\langle 0, \delta)$ such that $\gamma(0) = \mathring{X}$, $\gamma(t) \in U$ for $t \in (0, \delta)$ and the limit of $\xi_1(\gamma(t))$ is infinite for $t \to 0_+$.*

Proof. Let us consider the vector $\mathring{X} = (\mathring{x}_1, \mathring{x}_2, \mathring{x}_3, \mathring{x}_4, \mathring{z}_1, \mathring{z}_2)$ which lies in V and not all $\mathring{x}_i$ are equal to zero. Let us consider the curves in $\mathfrak{m}$

$$\begin{aligned} \gamma_1 &= (\mathring{x}_1 + t^2, \mathring{x}_2, \mathring{x}_3, \mathring{x}_4, \mathring{z}_1 + t, \mathring{z}_2), \\ \gamma_2 &= (\mathring{x}_1 + t^2, \mathring{x}_2, \mathring{x}_3, \mathring{x}_4, \mathring{z}_1, \mathring{z}_2 + t). \end{aligned} \tag{17}$$

Now we calculate

$$\begin{aligned}\xi_1(\gamma_1(t)) &= \frac{2\varepsilon(\mathring{z}_1+t)((\mathring{x}_1+t^2)\mathring{x}_4+\mathring{x}_2\mathring{x}_3)+2\mathring{z}_2(-(\mathring{x}_1+t^2)\mathring{x}_3+\mathring{x}_2\mathring{x}_4)}{(\mathring{x}_1+t^2)^2+\mathring{x}_2^2-\mathring{x}_3^2-\mathring{x}_4^2}\\ &= \frac{2\varepsilon(\mathring{z}_1+t)(\mathring{x}_4t^2+\mathring{x}_1\mathring{x}_4+\mathring{x}_2\mathring{x}_3)+2\mathring{z}_2(-\mathring{x}_3t^2-\mathring{x}_1\mathring{x}_3+\mathring{x}_2\mathring{x}_4)}{t^4+2\mathring{x}_1t^2+\mathring{x}_1^2+\mathring{x}_2^2-\mathring{x}_3^2-\mathring{x}_4^2}\\ &= \frac{2\varepsilon t^3\mathring{x}_4+2t^2(\varepsilon\mathring{z}_1\mathring{x}_4-\mathring{z}_2\mathring{x}_3)+2\varepsilon t(\mathring{x}_1\mathring{x}_4+\mathring{x}_2\mathring{x}_3)+2(\varepsilon\mathring{z}_1(\mathring{x}_1\mathring{x}_4+\mathring{x}_2\mathring{x}_3)+\mathring{z}_2(-\mathring{x}_1\mathring{x}_3+\mathring{x}_2\mathring{x}_4))}{t^4+2\mathring{x}_1t^2}. \end{aligned} \tag{18}$$

In the same way we obtain

$$\begin{aligned}\xi_1(\gamma_2(t)) &= \frac{2\varepsilon\mathring{z}_1((\mathring{x}_1+t^2)\mathring{x}_4+\mathring{x}_2\mathring{x}_3)+2(\mathring{z}_2+t)(-(\mathring{x}_1+t^2)\mathring{x}_3+\mathring{x}_2\mathring{x}_4)}{(\mathring{x}_1+t^2)^2+\mathring{x}_2^2-\mathring{x}_3^2-\mathring{x}_4^2}\\ &= \frac{2\varepsilon\mathring{z}_1(\mathring{x}_4t^2+\mathring{x}_1\mathring{x}_4+\mathring{x}_2\mathring{x}_3)+2(\mathring{z}_2+t)(-\mathring{x}_3t^2-\mathring{x}_1\mathring{x}_3+\mathring{x}_2\mathring{x}_4)}{t^4+2\mathring{x}_1t^2+\mathring{x}_1^2+\mathring{x}_2^2-\mathring{x}_3^2-\mathring{x}_4^2}\\ &= \frac{-2t^3\mathring{x}_3+2t^2(\varepsilon\mathring{z}_1\mathring{x}_4-\mathring{z}_2\mathring{x}_3)+2t(-\mathring{x}_1\mathring{x}_3+\mathring{x}_2\mathring{x}_4)+2(\varepsilon\mathring{z}_1(\mathring{x}_1\mathring{x}_4+\mathring{x}_2\mathring{x}_3)+\mathring{z}_2(-\mathring{x}_1\mathring{x}_3+\mathring{x}_2\mathring{x}_4))}{t^4+2\mathring{x}_1t^2}. \end{aligned} \tag{19}$$

Let us notice that the coefficients of t^1 in the numerators of the last fractions in (18) and (19) are the polynomials $\mathring{k}_2, \mathring{k}_3$ defined by the analogs of formulas (11). We denote the coefficient of t^0 (which is the same in the numerators of (18) and (19)) by $\mathring{k}_1$.

Now, if $\mathring{k}_1 \neq 0$ (which is equivalent to $\mathring{X} \in V_0$), we have $\lim_{t\to 0}\xi_1(\gamma_1(t)) = \operatorname{sgn}(\mathring{k}_1/\mathring{x}_1)\cdot\infty$ and $\lim_{t\to 0}\xi_1(\gamma_2(t)) = \operatorname{sgn}(\mathring{k}_1/\mathring{x}_1)\cdot\infty$.

If $\mathring{k}_1 = 0$ and either $\mathring{k}_2 \neq 0$ or $\mathring{k}_3 \neq 0$, we have either $\lim_{t\to 0+}\xi_1(\gamma_1(t)) = \operatorname{sgn}(\varepsilon\mathring{k}_2/\mathring{x}_1)\cdot\infty$ or $\lim_{t\to 0+}\xi_1(\gamma_2(t)) = \operatorname{sgn}(-\mathring{k}_3/\mathring{x}_1)\cdot\infty$. From the previous section we know that we cannot have $\mathring{k}_2 = \mathring{k}_3 = 0$ unless all $\mathring{x}_i$ are zero.

Thus, with the exception of for this special situation, we have proved the last part of Theorem 2.3. This includes also the case $z_1 = z_2 = 0$.

It remains to investigate the case $\mathring{x}_1 = \mathring{x}_2 = \mathring{x}_3 = \mathring{x}_4 = 0$. Let us consider the curve

$$\gamma(t) = (\sqrt{t}+t^2, 0, \sqrt{t}-t^2, 0, \mathring{z}_1, \mathring{z}_2 + 2t\sqrt[4]{t}), \tag{20}$$

for $t \geq 0$. We calculate

$$\begin{aligned}\xi_1(\gamma(t)) &= \frac{2\varepsilon\mathring{z}_1\cdot 0 - 2(\mathring{z}_2+2t\sqrt[4]{t})(\sqrt{t}+t^2)(\sqrt{t}-t^2)}{(\sqrt{t}+t^2)^2-(\sqrt{t}-t^2)^2}\\ &= \frac{4t^{21/4}+2\mathring{z}_2t^4-4t^{9/4}-2\mathring{z}_2t}{4t^{5/2}} \end{aligned} \tag{21}$$

and we see that $\lim_{t\to 0+}\xi_1(\gamma(t)) = -\infty$. In particular, this limit is infinite at the origin $o \in \mathfrak{m}$. □

2.5. *Limits along curves in the null-cone*

In Section 2.4 we have investigated the limits of ξ_1 along curves in U approaching a vector $\mathring{X} \in V$. In general, the vector $\mathring{X}$ was not a null vector and the curves were also arbitrary. The idea of P. Meessen (who gave us just one special example) was to find an infinite limit of $\xi(X)$ for the null vector X along a curve which belongs to the intersection $N \cap U$. We are going to do such a construction in general now.

Theorem 2.4. *For any vector $\mathring{X} \in N \cap V$, there is a curve $\gamma(t)$ with the values in $\mathfrak{m}$ and defined on an interval $\langle 0, \delta)$ such that $\gamma(0) = \mathring{X}$, $\gamma(t) \in N \cap U$ for $t \in (0, \delta)$ and the limit of $\xi_1(\gamma(t))$ is infinite for $t \to 0_+$.*

Proof. Because the null-cone N depends on the metric (see Theorem 2.2), we construct the curves for each case $\varepsilon = \pm 1$ separately.

Case $\varepsilon = 1$:

Let the vector $\mathring{X} = (\mathring{x}_1, \mathring{x}_2, \mathring{x}_3, \mathring{x}_4, 0, 0)$ be null and lying in V. Let us consider the curves

$$\gamma_1(t) = \left(\mathring{x}_1 + at^2, \mathring{x}_2 + bt^2, \mathring{x}_3 - ct^2, \mathring{x}_4 - dt^2, 0, t\sqrt{2(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)}\right)$$
$$\gamma_2(t) = \left(\mathring{x}_1 + at^2, \mathring{x}_2 + bt^2, \mathring{x}_3 - ct^2, \mathring{x}_4 - dt^2, t\sqrt{2(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)}, 0\right), \tag{22}$$

where a, b, c, d are the parameters such that $a = \cos(\alpha)$, $b = \sin(\alpha)$, $c = \cos(\beta)$, $d = \sin(\beta)$ for some α, β. In this way we have $a^2 + b^2 - c^2 - d^2 = 0$ and both these curves lie in N. We calculate

$$\begin{aligned}
&\xi_1(\gamma_1(t)) \\
&\quad = \frac{2t\sqrt{2(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)}\left(-(\mathring{x}_1 + at^2)(\mathring{x}_3 - ct^2) + (\mathring{x}_2 + bt^2)(\mathring{x}_4 - dt^2)\right)}{(\mathring{x}_1 + at^2)^2 + (\mathring{x}_2 + bt^2)^2 - (\mathring{x}_3 - ct^2)^2 - (\mathring{x}_4 - dt^2)^2} \\
&\quad = \frac{2t\sqrt{2(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)}\left(-\mathring{x}_1\mathring{x}_3 + \mathring{x}_2\mathring{x}_4 + t^2(c\mathring{x}_1 - d\mathring{x}_2 - a\mathring{x}_3 + b\mathring{x}_4) + t^4(ac - bd)\right)}{\mathring{x}_1^2 + \mathring{x}_2^2 - \mathring{x}_3^2 - \mathring{x}_4^2 + 2t^2(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4) + t^4(a^2 + b^2 - c^2 - d^2)} \\
&\quad = \frac{\sqrt{2(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)}\left((-\mathring{x}_1\mathring{x}_3 + \mathring{x}_2\mathring{x}_4) + t^2(c\mathring{x}_1 - d\mathring{x}_2 - a\mathring{x}_3 + b\mathring{x}_4) + t^4(ad - bd)\right)}{t(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)}.
\end{aligned} \tag{23}$$

If not all $\mathring{x}_i$ are equal to zero, we can choose the parameters α, β such that $a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4 \neq 0$. Then, if $\mathring{k}_3 \neq 0$, the limit of $\xi_1(\gamma(t))$ for $t \to 0$ is infinite. In the same way we obtain infinite limit for the curve γ_2 and for $\mathring{k}_2 \neq 0$. Again, we cannot have $\mathring{k}_2 = \mathring{k}_3 = 0$.

In the case $\mathring{x}_1 = \mathring{x}_2 = \mathring{x}_3 = \mathring{x}_4 = 0$, the only interesting point is the origin (other points do not lie in N). We make a modification of the above curve by choosing $\mathring{x}_2 = \mathring{x}_4 = b = d = 0$, $a = c = 1$ and by putting $\mathring{x}_1 = \mathring{x}_3 = \sqrt{t}$. We obtain the curve $\gamma(t)$ (for $t \geq 0$) described (for general $\mathring{z}_1, \mathring{z}_2$) by the formula (20) in Section 2.4. For $\mathring{z}_1 = \mathring{z}_2 = 0$ this curve lies in N and the limit of $\xi_1(\gamma(t))$ for $t \to 0_+$ is also infinite.

Case $\varepsilon = -1$:

Let the vector $\mathring{X} = (\mathring{x}_1, \mathring{x}_2, \mathring{x}_3, \mathring{x}_4, \mathring{z}_1, \mathring{z}_2)$ be in $N \cap V$, let $\mathring{z}_2 = \varepsilon'\mathring{z}_1$ (where $\varepsilon' = \pm 1$) and let us consider the curve

$$\begin{aligned}
\gamma(t) = (&\mathring{x}_1 - 2a\mathring{z}_1 t, \mathring{x}_2 - 2b\mathring{z}_1 t, \mathring{x}_3 + 2c\mathring{z}_1 t, \mathring{x}_4 + 2d\mathring{z}_1 t, \\
&\mathring{z}_1 + (a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)t, \varepsilon'\mathring{z}_1 - \varepsilon'(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)t\big),
\end{aligned} \tag{24}$$

where the parameters a, b, c, d are as in the case $\varepsilon = 1$. Again, $a^2 + b^2 - c^2 - d^2 = 0$ and the curve lies in N. We shall calculate the limit for $t \to 0_+$ of the function

$$\xi_1(\gamma(t)) = \frac{P(t)}{Q(t)}, \tag{25}$$

where

$$\begin{aligned}
P(t) &= -2\left(\mathring{z}_1 + (a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)t\right) \cdot \left((\mathring{x}_1 - 2a\mathring{z}_1 t)(\mathring{x}_4 + 2d\mathring{z}_1 t) + (\mathring{x}_2 - 2b\mathring{z}_1 t)(\mathring{x}_3 + 2c\mathring{z}_1 t)\right) \\
&\quad + 2\left(\varepsilon'\mathring{z}_1 - \varepsilon'(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)t\right) \\
&\quad \cdot \left(-(\mathring{x}_1 - 2a\mathring{z}_1 t)(\mathring{x}_3 + 2c\mathring{z}_1 t) + (\mathring{x}_2 - 2b\mathring{z}_1 t)(\mathring{x}_4 + 2d\mathring{z}_1 t)\right) \\
&= -2\left(\mathring{z}_1 + (a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)t\right)\left(\mathring{x}_1\mathring{x}_4 + \mathring{x}_2\mathring{x}_3 + O(t)\right) \\
&\quad + 2\left(\varepsilon'\mathring{z}_1 - \varepsilon'(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)t\right)\left(-\mathring{x}_1\mathring{x}_3 + \mathring{x}_2\mathring{x}_4 + O(t)\right), \\
Q(t) &= (\mathring{x}_1 - 2a\mathring{z}_1 t)^2 + (\mathring{x}_2 - 2b\mathring{z}_1 t)^2 - (\mathring{x}_3 + 2c\mathring{z}_1 t)^2 - (\mathring{x}_4 + 2d\mathring{z}_1 t)^2 \\
&= 4\mathring{z}_1^2(a^2 + b^2 - c^2 - d^2)t^2 - 4\mathring{z}_1(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)t + \mathring{x}_1^2 + \mathring{x}_2^2 - \mathring{x}_3^2 - \mathring{x}_4^2 \\
&= -4\mathring{z}_1(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)t.
\end{aligned} \tag{26}$$

We see easily that the constant term in $P(t)$ is $\mathring{k}_1 = -\mathring{z}_1(\mathring{x}_1\mathring{x}_4 + \mathring{x}_2\mathring{x}_3) - \varepsilon'\mathring{z}_1(\mathring{x}_1\mathring{x}_3 - \mathring{x}_2\mathring{x}_4)$. Again, we can always choose the parameters a, b, c, d such that $a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4 \neq 0$. Hence, $Q(t)$ is a nonzero multiple of t and the limit of $\xi_1(\gamma(t))$ for $t \to 0_+$ is infinite if $\mathring{k}_1 \neq 0$. This is the case $\mathring{X} \in N \cap V_0$.

Now, let us describe the set $N \cap V_1$. We first suppose not all x_i equal to zero and not all z_j equal to zero. Here the three equations

$$-x_1^2 - x_2^2 + x_3^2 + x_4^2 - z_1^2 + z_2^2 = 0,$$

$$-x_1^2 - x_2^2 + x_3^2 + x_4^2 = 0,$$
$$z_1(x_1x_4 + x_2x_3) + z_2(x_1x_3 - x_2x_4) = 0 \tag{27}$$

must be satisfied. From the first two equations we obtain immediately $z_2 = \varepsilon' z_1$, for $\varepsilon' = \pm 1$. Then a longer but routine calculation shows that the second and the third equation of (27) are equivalent to

$$x_1 = \pm\frac{\sqrt{2}}{2}(x_4 - \varepsilon' x_3),$$
$$x_2 = \pm\frac{\sqrt{2}}{2}(x_3 + \varepsilon' x_4). \tag{28}$$

where either the plus signs (or the minus signs) are valid in both formulas (28).

We will consider the case when $\varepsilon' = 1$ and both signs in (28) are positive. The other combinations of signs are treated analogously. Let the vector $\mathring{X} \in N \cap V_1$ have the coordinates

$$\mathring{X} = \left(\frac{\sqrt{2}}{2}(\mathring{x}_4 - \mathring{x}_3), \frac{\sqrt{2}}{2}(\mathring{x}_4 + \mathring{x}_3), \mathring{x}_3, \mathring{x}_4, \mathring{z}_1, \mathring{z}_1\right). \tag{29}$$

We consider the curve $\gamma(t)$ for $t \geq 0$, starting at $\mathring{X}$ (for $t = 0$) and given by

$$\gamma(t) = \left(\frac{\sqrt{2}}{2}(\mathring{x}_4 - \mathring{x}_3) - t, \gamma_2(t), \mathring{x}_3, \mathring{x}_4, \mathring{z}_1, \mathring{z}_1 + t^2\right), \tag{30}$$

where $\gamma_2(t) = \sqrt{\frac{1}{2}(\mathring{x}_4 + \mathring{x}_3)^2 + \sqrt{2}(\mathring{x}_4 - \mathring{x}_3)t + (2\mathring{z}_1 - 1)t^2 + t^4}$. This expression is the consequence of the requirement $\gamma(t) \in N$. We check easily that the curve γ lies in the null-cone N and for $t > 0$ it lies in U. We calculate

$$\xi_1(\gamma(t)) = \frac{P(t)}{Q(t)}, \tag{31}$$

where

$$\begin{aligned} P(t) &= -2\mathring{z}_1\left(\left(\frac{\sqrt{2}}{2}(\mathring{x}_4 - \mathring{x}_3) - t\right)\mathring{x}_4 + \gamma_2(t)\mathring{x}_3\right) \\ &\quad + 2(\mathring{z}_1 + t^2)\left(-\left(\frac{\sqrt{2}}{2}(\mathring{x}_4 - \mathring{x}_3) - t\right)\mathring{x}_3 + \gamma_2(t)\mathring{x}_4\right) \\ &= 2\mathring{x}_3t^3 + \left(\sqrt{2}\mathring{x}_3(\mathring{x}_3 - \mathring{x}_4) + 2\mathring{x}_4\gamma_2(t)\right)t^2 + 2\mathring{z}_1(\mathring{x}_3 + \mathring{x}_4)t + \sqrt{2}\mathring{z}_1(\mathring{x}_3^2 - \mathring{x}_4^2) + 2\mathring{z}_1(\mathring{x}_4 - \mathring{x}_3)\gamma_2(t), \end{aligned}$$
$$Q(t) = \left(\frac{\sqrt{2}}{2}(\mathring{x}_4 - \mathring{x}_3) - t\right)^2 + \gamma_2(t)^2 - \mathring{x}_3^2 - \mathring{x}_4^2 = 2\mathring{z}_1t^2 + t^4. \tag{32}$$

We obtain

$$\begin{aligned} \lim_{t\to 0+}\xi_1(\gamma(t)) &= \lim_{t\to 0+}\frac{2\mathring{x}_3t + \left(\sqrt{2}\mathring{x}_3(\mathring{x}_3 - \mathring{x}_4) + 2\mathring{x}_4\gamma_2(t)\right)}{2\mathring{z}_1 + t^2} \\ &\quad + \lim_{t\to 0+}\frac{2\mathring{z}_1(\mathring{x}_3 + \mathring{x}_4)t + \sqrt{2}\mathring{z}_1(\mathring{x}_3^2 - \mathring{x}_4^2) + 2\mathring{z}_1(\mathring{x}_4 - \mathring{x}_3)\gamma_2(t)}{2\mathring{z}_1t^2 + t^4}. \end{aligned} \tag{33}$$

It is easy to see that the first limit on the right-hand side is finite. By using the l'Hospital's rule, we easily verify that the second limit is equal to ∞ for $\mathring{x}_3 > -\mathring{x}_4$ and for $\mathring{x}_3 = -\mathring{x}_4, \mathring{x}_3 < 0$. For $\mathring{x}_3 < -\mathring{x}_4$ and for $\mathring{x}_3 = -\mathring{x}_4, \mathring{x}_3 > 0$, we can consider the curve

$$\gamma(t) = \left(\frac{\sqrt{2}}{2}(\mathring{x}_4 - \mathring{x}_3) + t, \gamma_2(t), \mathring{x}_3, \mathring{x}_4, \mathring{z}_1, \mathring{z}_1 + t^2\right), \tag{34}$$

where $\gamma_2(t) = -\sqrt{\frac{1}{2}(\mathring{x}_4 + \mathring{x}_3)^2 - \sqrt{2}(\mathring{x}_4 - \mathring{x}_3)t + (2\mathring{z}_1 - 1)t^2 + t^4}$. Again, the limit (analogous to the second limit in the formula (33)) is equal to ∞.

Now, we consider the case $\mathring{z}_1 = \mathring{z}_2 = 0$ (and not all x_i equal to zero). The following curves (the modifications of the curves given by the formulas (22) in the case $\varepsilon = 1$, where the parameters a, b, c, d are as before) lie in N:

$$\gamma_1(t) = \left(\mathring{x}_1 + at^2, \mathring{x}_2 + bt^2, \mathring{x}_3 - ct^2, \mathring{x}_4 - dt^2, 0, t\sqrt{2(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)}\right)$$

$$\gamma_2(t) = \left(\mathring{x}_1 - at^2, \mathring{x}_2 - bt^2, \mathring{x}_3 + ct^2, \mathring{x}_4 + dt^2, t\sqrt{2(a\mathring{x}_1 + b\mathring{x}_2 + c\mathring{x}_3 + d\mathring{x}_4)}, 0\right). \tag{35}$$

The fact that, for $t \to 0_+$, the limit along $\gamma_1(t)$ for $\mathring{k}_3 \neq 0$, and the limit along $\gamma_2(t)$ for $\mathring{k}_2 \neq 0$ are infinite, is an easy modification of the calculations for the case $\varepsilon = 1$ (formulas (22)).

Finally, we describe the situation $\mathring{x}_1 = \mathring{x}_2 = \mathring{x}_3 = \mathring{x}_4 = 0$. We modify the curve given by the formula (24) by choosing $b = c = \mathring{x}_2 = \mathring{x}_3 = 0$, $a = d = \frac{1}{2}$ and by putting $\mathring{x}_1 = \mathring{x}_4 = \sqrt{t}$. We obtain the curve

$$\gamma(t) = \left(\sqrt{t} - \mathring{z}_1 t, 0, 0, \sqrt{t} + \mathring{z}_1 t, \mathring{z}_1 + t^{3/2}, \varepsilon' \mathring{z}_1 - \varepsilon' t^{3/2}\right) \tag{36}$$

which lies in N and we calculate

$$\xi_1(\gamma(t)) = \frac{-2(\mathring{z}_1 + t^{3/2})(\sqrt{t} - \mathring{z}_1 t)(\sqrt{t} + \mathring{z}_1 t)}{(\sqrt{t} - \mathring{z}_1 t)^2 - (\sqrt{t} + \mathring{z}_1 t)^2} = \frac{2\mathring{z}_1^2 t^{7/2} - 2t^{5/2} + 2\mathring{z}_1^3 t^2 - 2\mathring{z}_1 t}{-4\mathring{z}_1 t^{3/2}}. \tag{37}$$

It is easy to see that $\lim_{t\to 0+} \xi_1(\gamma(t)) = \infty$ for $\mathring{z}_1 \neq 0$.

If, moreover, $\mathring{z}_1 = \mathring{z}_2 = 0$, we can modify this curve by putting $\mathring{z}_1 = t$, $\mathring{z}_2 = \varepsilon' t$ and consider the curve

$$\gamma(t) = \left(\sqrt{t} - t^2, 0, 0, \sqrt{t} + t^2, t + t^{3/2}, \varepsilon'(t - t^{3/2})\right). \tag{38}$$

This curve belong to the null-cone N, it starts at the origin and the limit of $\xi_1(\gamma(t))$ for $t \to 0_+$ is infinite. □

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References

[1] G. Calvaruso, R. Marinosci, Homogeneous geodesics of three-dimensional unimodular Lorentzian Lie groups Mediterranean, J. Math. 3 (3–4) (2006) 467–481.
[2] Z. Dušek, Explicit geodesic graphs on some H-type groups, Rend. Circ. Mat. Palermo (2) Suppl. 69 (2002) 77–88.
[3] Z. Dušek, O Kowalski, Geodesic graphs on the 13-dimensional group of Heisenberg type, Math. Nachr. 254–255 (2003) 87–96.
[4] Z. Dušek, O. Kowalski, Light-like homogeneous geodesics and the Geodesic Lemma for any signature, Publ. Math. Debrecen 71 (1–2) (2007) 245–252.
[5] Z. Dušek, O. Kowalski, Examples of pseudo-Riemannian g.o. manifolds, in: I. Mladenov, M. de Leon (Eds.), Proc. 8th Int. Conf. Varna 2006, 2007, pp. 145–155.
[6] Z. Dušek, O. Kowalski, S. Nikčević, New examples of Riemannian g.o. manifolds in dimension 7, Differential Geom. Appl. 21 (2004) 65–78.
[7] J. Figueroa-O'Farrill, P. Meessen, S. Philip, Homogeneity and plane-wave limits, J. High Energy Phys. 05 (2005) 050.
[8] C. Gordon, Homogeneous Riemannian manifolds whose geodesics are orbits, Progr. Nonlinear Differential Equations Appl. 20 (1996) 155–174.
[9] A. Kaplan, On the geometry of groups of Heisenberg type, Bull. London Math. Soc. 15 (1983) 35–42.
[10] O. Kowalski, S. Nikčević, On geodesic graphs of Riemannian g.o. spaces, Arch. Math. 73 (1999) 223–234. Appendix: Arch. Math. 79 (2002) 158–160.
[11] O. Kowalski, J. Szenthe, On the existence of homogeneous geodesics in homogeneous Riemannian manifolds, Geom. Dedicata 81 (2000) 209–214; Geom. Dedicata 84 (2001) 331–332 (Erratum).
[12] O. Kowalski, L. Vanhecke, Riemannian manifolds with homogeneous geodesics, Boll. Unione Mat. Ital. B(7) 5 (1991) 189–246.
[13] O. Kowalski, Z. Vlášek, Homogeneous Riemannian manifolds with only one homogeneous geodesic, Publ. Math. Debrecen 62 (3–4) (2003) 437–446.
[14] P. Meessen, Homogeneous Lorentzian spaces whose null-geodesics are canonically homogeneous, Lett. Math. Phys. 75 (2006) 209–212.
[15] S. Philip, Penrose limits of homogeneous spaces, J. Geom. Phys. 56 (2006) 1516–1533.
[16] J. Szenthe, Sur la connection naturelle à torsion nulle, Acta Sci. Math. (Szeged) 38 (1976) 383–398.